

## Coordinate Stretching and Interface Location I. Star Models in Quasi-Static Equilibrium<sup>1</sup>

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### ABSTRACT

The use of the coordinate stretching procedure of Poincaré and Lighthill is investigated for its applicability to the question of interface location in two-point boundary-value problems. The case in which an interface separates zones of convective and radiative equilibrium in star models is used to exemplify the technique. We discuss briefly the necessary conditions for choosing the arbitrary Poincaré-Lighthill function, and we formulate the linearized problem by making a particular choice for the function.

Most existing algorithms for the solution of nonlinear equations with two-point boundary conditions involve the solution of the linearized equations obtained by perturbing the four dependent variables and iterating until convergence is reached. The purpose of this note is to point out the role of the coordinate stretching technique of Poincaré [1] and Lighthill [2] in helping to locate interfaces between the two boundaries. We exemplify the technique by considering the problem of a spherical star in quasi-static equilibrium, thereby confining the coordinate stretching to the independent space-variable. The question of the stretching of the time scale will be taken up at a later stage.

Consider the system of nonlinear equations governing the structure of a star in radiative equilibrium:

$$dy^i/dx = f^i(y, x) \quad (i = 1, 2, 3, 4), \quad (1)$$

where  $y = \{y^1, y^2, y^3, y^4\}$ . We identify  $y^4$  with the temperature and  $x$  with the mass fraction and we assume that the range of  $x$  has been normalized to unity. When the temperature gradient  $dy^4/dx$  equals the adiabatic gradient; i.e.,

$$\frac{dy^4}{dx} = \left(\frac{dy^4}{dx}\right)_{\text{ad}}, \quad (2)$$

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convection ensues, and the fourth component of Eq. (1) must be replaced by the corresponding adiabatic gradient

$$dy^4/dx = \tilde{f}^4(y, x). \quad (3)$$

For simplicity we assume that the star is chemically homogeneous. The case for which a discontinuity in chemical composition occurs at the interface can be generalized from the present work without difficulty.

The standard perturbation procedure is to expand  $y$  about a trial solution  $y_0$ ,

$$y = y_0 + y_1, \quad (4)$$

and assume that  $y_1 \ll y_0$ . The Poincaré–Lighthill (or PL) expansion generalizes the procedure by expanding the independent variable as well:

$$x = x_0 + x_1, \quad (5)$$

where  $x_1$  is arbitrary but small and where now all functions  $y_0, y_1, x_1$  are functions of  $x_0$ . Substituting expansions (4) and (5) into Eq. (1) and neglecting all quantities higher than first order, we obtain

$$\frac{dy_1}{dx_0} = y_1^i \left( \frac{\partial f}{\partial y^i} \right)_0 + x_1 \left( \frac{\partial f}{\partial x} \right)_0 + \frac{dx_1}{dx_0} f_0 + \left( f - \frac{dy}{dx} \right)_0, \quad (6)$$

where  $f_0 \equiv f(y_0, x_0)$ . In Eq. (6) all terms with zero subscripts are evaluated from the trial solution and we retain the zeroth-order quantities

$$E_0 = (f - dy/dx)_0,$$

which are the errors in the gradients of the trial solution; summation occurs over repeated indexes. Whenever the Schwarzschild criterion (2) is satisfied by the trial solution, i.e., when

$$\left( \frac{dy^4}{dx} \right)_0 = \left( \frac{dy^4}{dx} \right)_{0,ad}, \quad (7)$$

convection is the dominant mode of energy transport and  $f^4$  in Eq. (6) must be replaced by  $\tilde{f}^4$  according to Eq. (3).

We may now use the freedom of choice given by the arbitrary PL function  $x_1(x_0)$  to assist in locating the interface. Clearly,  $x_1(x_0)$  must comply with the conditions pertaining at the interface. These are (in the present case) the stability criterion (2) which from Eqs. (1) and (3) we rewrite as

$$S(y, x) = f^4(y, x) - \tilde{f}^4(y, x). \quad (8)$$

At the interface  $S = 0$ ; similar criteria can be established for other physical conditions. Performing the PL expansion on  $S(y, x)$  and neglecting terms of order higher than 1 as before, we find:

$$S(y, x) = S_0 + y_1^i \left( \frac{\partial S}{\partial y^i} \right)_0 + x_1 \left( \frac{\partial S}{\partial x} \right)_0, \quad (9)$$

where  $S_0 \equiv S(y_0, x_0)$  is the value of  $S$  obtained from the trial solution. At the trial interface,  $S_0 = 0$ . The criterion for the determination of  $x_1(x_0)$  is that  $S$  shall be zero at the corrected interface as well. This defines  $x_1(x_0)$  at the interface and hence, from Eq. (5), the corrected position of the interface. Thus, denoting by bars the quantities evaluated at the interface, we have from Eq. (9),

$$\bar{x}_1(\bar{x}_0) = - \left( \frac{\partial \bar{S}}{\partial \bar{x}} \right)_0^{-1} \left[ \bar{y}_1^i \left( \frac{\partial \bar{S}}{\partial \bar{y}^i} \right)_0 \right]. \quad (10)$$

We now choose the arbitrary function  $x_1(x_0)$  to be such that the size of the range of  $x$  is unaltered after an iteration (since the mass of the star is specified). Thus

$$x_1(0) = 0, \quad (11)$$

$$x_1(1) = 0. \quad (12)$$

In addition  $x_1(x_0)$  must have the value  $\bar{x}_1$  at  $\bar{x}_0$ , according to Eq. (10). These conditions are not sufficient to determine  $x_1(x_0)$  completely, and we therefore can arbitrarily select other criteria. For example, we might choose for  $x_1(x_0)$  the function that maximizes the rate of convergence to the true solution. For present purposes, however, it is sufficient to complete the formulation in a strictly linear fashion and choose  $x_1(x_0)$  to be linear. In view of Eqs. (10)–(12),  $x_1(x_0)$  cannot therefore be continuous and smooth. Hence we let

$$x_1 = \left( \frac{\bar{x}_1}{\bar{x}_0} \right) x_0, \quad (0 \leq x_0 \leq \bar{x}_0), \quad (13)$$

$$x_1 = \left( \frac{\bar{x}_1}{\bar{x}_0 - 1} \right) (x_0 - 1), \quad (\bar{x}_0 \leq x_0 \leq 1). \quad (14)$$

A simple visualization of this particular stretching procedure is achieved by regarding the range ( $0 \leq x \leq 1$ ) as an elastic rod obeying Hooke's law in compression and elongation. We regard  $x_0$  as a Lagrangian variable which moves with the material of the rod during elastic deformations. Once the point  $\bar{x}_0$  is located from the Schwarzschild criterion, changes in the position of the trial interface will stretch the rod in one region while simultaneously compressing it in the other. Whether either region suffers a stretching or a compression depends upon the sign

of  $\bar{x}_1$ . The procedure is repeated on successive iterations until convergence, at which the final value  $\bar{x} = \bar{x}_0 + \bar{x}_1$  gives the position of the interface. More involved choices can be made when there is more than one interface, and laws of elasticity more complicated than Hooke's law can be used in the stretching of the  $x$  coordinate.

By use of Eqs. (13) and (14), Eq. (6) becomes

$$\frac{dy_1}{dx_0} = y_1^i \left( \frac{\partial f}{\partial y^i} \right)_0 + \left( \frac{\bar{x}_1}{\bar{x}_0} \right) \left[ x_0 \left( \frac{\partial f}{\partial x} \right)_0 + f_0 \right] + E_0 \quad (0 \leq x_0 \leq \bar{x}_0), \quad (15)$$

$$\frac{dy_1}{dx_0} = y_1^i \left( \frac{\partial f}{\partial y^i} \right)_0 + \left( \frac{\bar{x}_1}{\bar{x}_0 - 1} \right) \left[ (x_0 - 1) \left( \frac{\partial f}{\partial x} \right)_0 + f_0 \right] + E_0 \quad (\bar{x}_0 \leq x_0 \leq 1), \quad (16)$$

where  $\bar{x}_1$  must be obtained from Eq. (10). The solution of these equations may be organized according to existing techniques (see, e.g., [3]–[5]). Computational experimentation on this and related problems is underway.

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